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Taming the Leibniz rule on the lattice

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Abstract: We study a product rule and a difference operator equipped with Leibniz rule in a general framework of lattice field theory. It is shown that the difference operator can be determined by the product rule and some initial data through the Leibniz rule. This observation leads to a no-go theorem that it is impossible to construct any difference operator and product rule on a lattice with the properties of (i) translation invariance, (ii) locality and (iii) Leibniz rule. We present a formalism to overcome the difficulty by an infinite flavor extension or a matrix expression of a lattice field theory.

Keywords: Lattice Quantum Field Theory, Lattice Gauge Field Theories.

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Contents

1. Introduction

It is of great importance to formulate supersymmetric theories on a lattice to study non-perturbative dynamics, especially supersymmetry breaking. Over the last thirty years, a considerable number of attempts have been made to construct lattice supersymmetric models [1 − 3]. However, none of them have not fully succeeded in realizing supersymmetry on a lattice. A key to construct interacting supersymmetric theories is to keep a Leibniz rule on a lattice [4, 5].[1] Naive difference operators, like forward/backward/symmetric ones, do not satisfy a Leibniz rule. Lattice models equipped with a Leibniz rule exist by allowing the non-locality of interactions or difference operators [4, 10 − 13]. In fact, it is pointed out that it is difficult to impose simultaneously the following three properties: (i) translation invariance, (ii) locality and (iii) Leibniz rule in any lattice field theories [14].[2]

In this article, we prove the above statement as a no-go theorem in general lattice theories. The requirement of the associative law leads us to an easier proof of the no-go theorem, but it is not necessary to the proof. We further show that it is impossible to solve the Leibniz rule problem even if a product rule of fields and a difference operator are extended to include multi-flavor indices. Our proof shows that a difference operator can be determined from information of a product rule and some initial data through the

[1] Recently, novel ideas of the noncommutativity approach [6] and the link approach [7] have been proposed to restore a Leibniz rule for supersymmetry transformations in twisted supersymmetric models on the lattice. Further investigation, however, seems to be necessary [8, 9].

[2] A no-go theorem in a restricted case was given in ref. [10, 13].

equation derived from a Leibniz rule. Then, it turns out that any local product rule inevitably leads to a non-local difference operator in any translationally invariant lattice theories of finite flavors. One way to escape from the no-go theorem is to introduce an infinite number of flavors and a nontrivial connection between lattice sites and flavors. We propose a translationally invariant local lattice theory that a difference operator satisfying a Leibniz rule is realized with a product rule equipped with an associative law in a matrix formulation.

In section 2, our fundamental tools of product rule, difference operator, translational invariance and locality are explained. In section 3, we see that the associative law restricts the form of the product rule essentially to a normal local product. We prove the no-go theorem for general one-flavor systems in section 4 and for general multi-flavor systems in section 5. In section 6, we present a lattice model that evades the no-go theorem by introducing an infinite number of flavors. Section 7 is devoted to summary and discussions.

2. Locality of product rule and difference operator on a lattice

A lattice gauge theory has usually treated only ultra local operators except Dirac operators such as Ginsparg-Wilson fermion or overlap-Dirac operator [15 − 17]. In order to analyze a Leibniz rule and an associative law on a lattice, we must generalize a product rule between fields and a difference operator on a field.

A lattice field product between ϕ_n and ψ_n is defined as

$$(\phi \cdot \psi)_n \equiv \sum_{l,m} C_{lmn} \phi_l \psi_m \,, \tag{2.1}$$

where the coefficient C_{lmn} becomes a key of this product definition keeping bi-linearity on both fields. The indices l, m, n imply positions on a lattice which has an infinite size. Although we restrict our consideration to one-dimensional lattice throughout this paper, the extension to higher dimensions will be straightforward. If one chooses

$$C_{lmn} = \delta_{l,n} \delta_{m,n} \,, \tag{2.2}$$

as the product rule, then it defines the normal product of lattice fields at the same point.

Another coefficient D on a field

$$(D\phi)_n \equiv \sum_m D_{mn} \phi_m \tag{2.3}$$

means a generalized difference operator keeping the linearity about the field. The difference operator for a constant field implies

$$\sum_m D_{mn} = 0 \,. \tag{2.4}$$

Two familiar examples for D_{mn} are the forward and backward difference operators defined, respectively, as

$$D^+_{mn} = \delta_{m,n+1} - \delta_{m,n} \,, \tag{2.5}$$

$$D^-_{mn} = \delta_{m,n} - \delta_{m,n-1} \,. \tag{2.6}$$

If the system has no external field, it should keep translational invariance. The invariance for C_{lmn} and D_{mn} is imposed as the following forms:

$$C_{lmn} = C(l-n, m-n)\,, \tag{2.7}$$
$$D_{mn} = D(m-n)\,. \tag{2.8}$$

The locality property is important in constructing local field theories after the continuum limit. To make the locality manifest, we define Fourier transform of the coefficients $C(k,l)$ and $D(m)$ by

$$\hat{C}(v,w) \equiv \sum_{k,l=-\infty}^{\infty} C(k,l) v^k w^l, \tag{2.9}$$
$$\hat{D}(z) \equiv \sum_{m=-\infty}^{\infty} D(m) z^m, \tag{2.10}$$

where v, w, z are S^1-variables given by $v = e^{ip},\ w = e^{iq},\ z = e^{ir}$ and will be extended to some complex domains later. As we will explain below, the locality of the product rule (2.1) and the difference operator (2.3) is directly related to the holomorphic property of the complex functions.

In terms of the complex function $\hat{D}(z)$, the condition (2.4) can be rewritten as

$$\hat{D}(1) = 0\,, \tag{2.11}$$

which may be regarded as an initial condition of the function $\hat{D}(z)$.

From our knowledge about lattice fields and complex analysis, the local property of the product rule and the difference operator restricts us to holomorphic functions for $\hat{C}(v,w)$ and $\hat{D}(z)$. To discuss more strictly, we prepare an annulus $\mathcal{D}_2 = \{(v,w) | 1-\epsilon < |v|, |w| < 1+\epsilon\}$ for $\hat{C}(v,w)$ and another annulus $\mathcal{D}_1 = \{z | 1-\epsilon < |z| < 1+\epsilon\}$ for $\hat{D}(z)$, where ϵ is a positive constant smaller than unity. The functions $\hat{C}(v,w)$ and $\hat{D}(z)$ are analytically extended to these annulus domains uniquely owing to their holomorphism. We state a lemma about the locality of $C(k,l)$ here.

Lemma 1. *The following two propositions are equivalent to each other:*

1. *A product rule $C(k,l)$ is local.*
2. *The corresponding $\hat{C}(v,w)$ is holomorphic on $\mathcal{D}_2$.*

If the proposition 1 holds, $C(k,l)$ is exponentially decaying as $o(\exp(-r_1|k|), \exp(-r_2|l|))$ for large $|k|$ and $|l|$ where r_1 and r_2 are some positive numbers. From this behavior of $C(k,l)$, we can define a complex function $\hat{C}(v,w)$ as

$$\hat{C}(v,w) \equiv \sum_{k,l} C(k,l) v^k w^l, \tag{2.12}$$

which is uniformly convergent in $\{(v,w) | e^{-r_1} < |v| < e^{r_1}, e^{-r_2} < |w| < e^{r_2}\}$ and is holomorphic on $\mathcal{D}_2$ where $1-\epsilon = e^{-r_1}$ for $r_1 < r_2$ ($1-\epsilon = e^{-r_2}$ for $r_2 < r_1$). Conversely,

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if the proposition 2 holds, eq. (2.12) shows the Laurent expansion with v and w, which converges on the annulus $\mathcal{D}_2$. The coefficient $C(k,l)$ for positive integers k,l behaves, with $0<\epsilon'<\epsilon$, as

$$|C(k,l)| = \left| \oint_{|v|=1+\epsilon'} \frac{dv}{2\pi i} \oint_{|w|=1+\epsilon'} \frac{dw}{2\pi i} \hat{C}(v,w) v^{-k-1} w^{-l-1} \right| \leq K_1 (1+\epsilon')^{-k-l}, \tag{2.13}$$

for negative integers k,l,

$$|C(k,l)| = \left| \oint_{|v|=1-\epsilon'} \frac{dv}{2\pi i} \oint_{|w|=1-\epsilon'} \frac{dw}{2\pi i} \hat{C}(v,w) v^{-k-1} w^{-l-1} \right| \leq K_2 (1-\epsilon')^{-k-l}, \tag{2.14}$$

for a positive k and a negative l,

$$|C(k,l)| = \left| \oint_{|v|=1+\epsilon'} \frac{dv}{2\pi i} \oint_{|w|=1-\epsilon'} \frac{dw}{2\pi i} \hat{C}(v,w) v^{-k-1} w^{-l-1} \right| \leq K_3 (1+\epsilon')^{-k} (1-\epsilon')^{-l}, \tag{2.15}$$

and a negative k and a positive l,

$$|C(k,l)| = \left| \oint_{|v|=1-\epsilon'} \frac{dv}{2\pi i} \oint_{|w|=1+\epsilon'} \frac{dw}{2\pi i} \hat{C}(v,w) v^{-k-1} w^{-l-1} \right| \leq K_4 (1-\epsilon')^{-k} (1+\epsilon')^{-l}, \tag{2.16}$$

where K_1, K_2, K_3 and K_4 are finite and positive constants because the absolute value of the holomorphic function $\hat{C}(v,w)$ is finite on $\mathcal{D}_2$. The relations (2.13)$\sim$(2.16) imply that $C(k,l)$ is decaying with $|k|$ and $|l|$ exponentially. Therefore, $C(k,l)$ is local.[3] In a similar way, we can show that if $D(m)$ is local, then the corresponding complex function $\hat{D}(z)$ is holomorphic on an annulus $\mathcal{D}_1$, and vice versa.

It is meaningful to define a terminology, "a trivial product rule", $\hat{C}(v,w)$ which is identically zero on the defined domain $\mathcal{D}_2$. Then, any field on a lattice multiplied by another field becomes vanishing and it is impossible to construct nontrivial theories. We can always find a 2-dimensional complex subdomain $\mathcal{F}_2 = \{(v,w)|\ \hat{C}(v,w) \neq 0\}$ in $\mathcal{D}_2$ in considering a nontrivial product rule $\hat{C}(v,w)$, otherwise the function $\hat{C}(v,w)$ is identically zero because of the identity theorem on complex functions.

3. Associative law and product rule on a lattice

In cases of interacting theories, we must consider field products of three-body or more. The consistency of field products in an actual model is often controlled by additional requirements like associativity. In this section, we examine the product rule that satisfies the associative law $(\phi\cdot\psi)\cdot\chi = \phi\cdot(\psi\cdot\chi)$. This can be read as

$$\sum_j C_{lmj} C_{jnk} = \sum_j C_{ljk} C_{mnj}\,. \tag{3.1}$$

[3]If we impose the smoothness on $\hat{C}(v,w)$ instead of the holomorphy, $C(l,k)$ is permitted to behave as power-damping.

After the translational invariance is imposed, the law can be rewritten, by use of the holomorphic function (2.12), as

$$\hat{C}(v,w)\hat{C}(vw,z) = \hat{C}(v,wz)\hat{C}(w,z)\,. \tag{3.2}$$

From eq. (3.2) and the holomorphy, it turns out that any nontrivial product rule $\hat{C}(v,w)$ can always be expressed, in terms of a holomorphic function $F(v)$ on $\mathcal{D}_1$, as

$$\hat{C}(v,w) = \frac{F(vw)}{F(v)F(w)}\,. \tag{3.3}$$

The proof is given in appendix.

To investigate the meaning of a factorization in eq. (3.3), we redefine a local field on a lattice as

$$\phi_n = \sum_m a_{mn}\phi'_m\,, \tag{3.4}$$

where the translational invariance and the locality are imposed, i.e. $a_{mn} = a(m-n)$ and $\hat{a}(v) = \sum_m a(m)v^m$. After the local redefinition (3.4) of fields, our product rule is transformed as

$$\hat{C}'(v,w) = \frac{\hat{a}(v)\hat{a}(w)}{\hat{a}(vw)}\hat{C}(v,w)\,. \tag{3.5}$$

This implies that we can always set $\hat{C}'(v,w) = 1$, by choosing $\hat{a}(v) = F(v)$, which is nothing but the normal local product $C'_{lmn} = \delta_{l,n}\delta_{m,n}$. Therefore, we conclude that the product rule satisfying the associative law (3.1) or (3.2) is essentially unique and is given by the normal product (2.2).

4. No-go theorem

In this section, we prove no-go theorems about a Leibniz rule on a lattice. We first assume the associative law for a product rule but later we give a proof without referring to the condition. The statement of the no-go theorem we first present is given as follows:

No-Go Theorem 1. *It is impossible to construct a lattice field theory in an infinite lattice volume with a nontrivial product rule (2.1) and a difference operator (2.3) that satisfy the following four properties: (i) translation invariance, (ii) locality, (iii) Leibniz rule and (iv) associative law.*

The proof is simple and goes as follows. A Leibniz rule $D(\phi\cdot\psi) = (D\phi)\cdot\psi + \phi\cdot(D\psi)$ can be translated into a relation between the product rule and the difference operator as

$$\sum_k C_{lmk}D_{kn} = \sum_k C_{kmn}D_{lk} + \sum_k C_{lkn}D_{mk}\,. \tag{4.1}$$

With the properties (i) and (ii), the relation (4.1) can be rewritten, in terms of the holomorphic functions $\hat{C}(v,w)$ and $\hat{D}(z)$ defined in eqs. (2.9) and (2.10), as

$$\hat{C}(v,w)\Big(\hat{D}(vw) - \hat{D}(v) - \hat{D}(w)\Big) = 0\,. \tag{4.2}$$

Since $\hat{C}(v,w)$ satisfying the associative law can be set to unity, as shown in the previous section, the condition (4.2) reduces to

$$\hat{D}(vw) - \hat{D}(v) - \hat{D}(w) = 0\,. \tag{4.3}$$

Differentiating it with respect to v and then putting $v = 1$, we have

$$w\partial_w \hat{D}(w) = \partial_v \hat{D}(v)|_{v=1}\,. \tag{4.4}$$

With the initial condition (2.11), the solution to eq. (4.4) is given by

$$\hat{D}(w) = \beta \log w\,, \tag{4.5}$$

where $\beta = \partial_v \hat{D}(v)|_{v=1}$. The coefficient β, however, has to vanish, otherwise the difference operator would become non-local because the logarithmic function $\log w$ is not holomorphic on $\mathcal{D}_1$. Hence, there is no nontrivial difference operator with the requirements (i)$\sim$(iv).

Without the associative law, the no-go theorem can still be proved as follows:

No-Go Theorem 2. *It is impossible to construct a lattice field theory in an infinite lattice volume with a nontrivial product rule (2.1) and a difference operator (2.3) that satisfy the following three properties: (i) translation invariance, (ii) locality and (iii) Leibniz rule.*

Without referring to the associative law, it is impossible, in general, to set $\hat{C}(v,w) = 1$, as discussed in the previous section. Instead, we use the existence of a domain $\mathcal{F}_2 = \{(v,w)|\ \hat{C}(v,w) \neq 0\}$, on which we have

$$\hat{D}(vw) - \hat{D}(v) - \hat{D}(w) = 0\,. \tag{4.6}$$

The domain $\mathcal{F}_2$ should span a 2-dimensional complex domain in $\mathcal{D}_2$, otherwise $\hat{C}(v,w)$ would be identically zero on $\mathcal{D}_2$. The general solution to eq. (4.6) is found as a logarithmic function on $\mathcal{F}_2$. The identity theorem enables to extend the domain $\mathcal{F}_2$ to $\mathcal{D}_2$. Thus, the difference operator cannot be holomorphic and hence is not local.

We would like to make some comments here. By remembering $w = e^{ip}$, the logarithmic function $\log w$ in (4.5) may be recognized as a SLAC-type derivative [10] in infinite systems. It is then interesting to note, as a corollary of our theorem, that a SLAC-type derivative must be adopted as the difference operator if we construct a lattice field theory satisfying a Leibniz rule with a local product rule. Although the proof is done in the case of an infinite lattice volume, the conclusion of the theorem can be kept even for general lattice theories with sufficiently large lattice size. We have proven the theorem for a one-dimensional theory and it is straightforward to generalize it for higher-dimensional cases.

5. Multi-flavor extension

We have presented a no-go theorem for general one-flavor systems. It is not difficult to extend it to N-flavor systems. A product rule and a difference operator are naturally extended as

$$(\phi \cdot \psi)_n^c \equiv \sum_{l,m} \sum_{a,b=1}^{N} C_{lmn}^{abc} \phi_l^a \psi_m^b \,, \tag{5.1}$$

$$(D\phi)_n^b \equiv \sum_{m} \sum_{a=1}^{N} D_{mn}^{ab} \phi_m^a \,, \tag{5.2}$$

where a, b, c denote flavor indices. A Leibniz rule in multi-flavor systems can be expressed as

$$\sum_{k} \sum_{d=1}^{N} C_{lmk}^{abd} D_{kn}^{dc} = \sum_{k} \sum_{d=1}^{N} C_{kmn}^{dbc} D_{lk}^{ad} + \sum_{k} \sum_{d=1}^{N} C_{lkn}^{adc} D_{mk}^{bd} \,. \tag{5.3}$$

The translation invariance and the locality for C_{lmn}^{abc} and D_{mn}^{ab} are defined in the same way as in single flavor systems, and lead to the holomorphic functions

$$\hat{C}^{abc}(v, w) \equiv \sum_{l,m} C^{abc}(l, m) v^l w^m, \tag{5.4}$$

$$\hat{D}^{ab}(z) \equiv \sum_{m} D^{ab}(m) z^m, \tag{5.5}$$

where $C_{lmn}^{abc} = C^{abc}(l - n, m - n)$ and $D_{mn}^{ab} = D^{ab}(m - n)$. If the difference operator D_{mn}^{ab} is independent of the flavor index, i.e.

$$D_{mn}^{ab} = \delta^{a,b} D_{mn} \,, \tag{5.6}$$

as in ordinary cases, then our no-go theorem holds exactly as before. Even in more general case of $D_{mn}^{ab} = \delta^{a,b} D_{mn}^{a}$, or equivalently

$$\hat{D}^{ab}(z) = \delta^{a,b} \hat{D}^{a}(z) \,, \tag{5.7}$$

our proof of the no-go theorem in the previous section can be applicable. For $\hat{D}^{ab}(z)$ to act properly on N-flavor fields, it is reasonable to assume that $\hat{D}^{ab}(z)$ can be diagonalized with respect to the flavor indices, as in eq. (5.7), by field redefinitions, which are identical to similarity transformations on $\hat{D}^{ab}(z)$. Thus, we succeeded in getting our no-go theorem for finite flavor systems.

No-Go Theorem 3. *It is impossible to construct a lattice field theory of finite flavors in an infinite lattice volume with a nontrivial product rule (5.1) and a difference operator (5.2) that satisfy the following three properties: (i) translation invariance, (ii) locality and (iii) Leibniz rule.*

Although a simple proof of the no-go theorem 3 was given above, it will be worth presenting another proof of the theorem that makes a connection clear between the product rule and the difference operator through the Leibniz rule. In terms of the holomorphic functions $\hat{C}^{abc}(v, w)$ and $\hat{D}^{ab}(z)$, the Leibniz rule (5.3) can be rewritten as

$$\sum_{d=1}^{N} \hat{C}^{abd}(v, w) \hat{D}^{dc}(vw) = \sum_{d=1}^{N} \hat{C}^{dbc}(v, w) \hat{D}^{ad}(v) + \sum_{d=1}^{N} \hat{C}^{adc}(v, w) \hat{D}^{bd}(w) \,. \tag{5.8}$$

It may be convenient, in the following analysis, to express $\hat{C}^{abc}(v,w)$ and $\hat{D}^{ab}(z)$ into $N\times N$ matrix forms as

$$\left(M^b(v,w)\right)_{ac} \equiv \hat{C}^{abc}(v,w)\,, \tag{5.9}$$

$$\left(D(z)\right)_{ab} \equiv \hat{D}^{ab}(z)\,. \tag{5.10}$$

In terms of them, the Leibniz rule (5.8) is further rewritten as

$$M^b(v,w)D(vw) = D(v)M^b(v,w) + \sum_{d=1}^{N} \hat{D}^{bd}(w)M^d(v,w)\,. \tag{5.11}$$

Setting $w=1$ in eq. (5.11) shows that $D(v)$ commutes with $M^b(v,1)$, i.e.

$$[M^b(v,1), D(v)] = 0 \qquad \text{for } b=1,2,\cdots,N\,, \tag{5.12}$$

where we have used

$$\hat{D}^{ab}(1) = 0 \;\text{ or }\; D(1) = 0 \tag{5.13}$$

which comes from the fact that $(D\phi)_n^b = 0$ for any constant field ϕ_m^a. Differentiating eq. (5.11) with respect to w and then setting $w=1$, we find

$$v\partial_v D(v) = -[M^b(v,1)^{-1}\partial_w M^b(v,w)|_{w=1},\, D(v)] + \sum_{d=1}^{N} \hat{D}'^{\,bd}(1)M^b(v,1)^{-1}M^d(v,1)\,, \tag{5.14}$$

where

$$\hat{D}'^{\,bd}(1) \equiv \partial_w \hat{D}^{bd}(w)|_{w=1}\,. \tag{5.15}$$

Here, we have used eqs. (5.12) and (5.13) and assumed the existence of $M^b(v,1)^{-1}$. We can regard eq. (5.14) as a differential equation for the difference operator $D(v)$ and formally solve it as

$$D(v) = \int_1^v \frac{dv'}{v'} U(v,v')\,X(v')\,U(v,v')^{-1}, \tag{5.16}$$

where

$$U(v,v') \equiv P\exp\left\{ -\int_{v'}^{v} \frac{dv''}{v''} M^b(v'',1)^{-1}\partial_w M^b(v'',w)|_{w=1} \right\}, \tag{5.17}$$

$$X(v') \equiv \sum_{d=1}^{N} \hat{D}'^{\,bd}(1)M^b(v',1)^{-1}M^d(v',1)\,. \tag{5.18}$$

Here, P denotes the path ordered product.

We should make several comments on eq. (5.16) here. It is interesting to note that the relation (5.16) implies that the difference operator can completely be determined from information about the product rule (more precisely, $M^b(v,1)$ and $\partial_w M^b(v,w)|_{w=1}$) and the initial values $\hat{D}'^{\,bd}(1)$. Although we have assumed the existence of $M^b(v,1)^{-1}$, this will, however, be assured for a nontrivial product rule because the function $\hat{C}^{abc}(v,w)$ (or $M^b(v,w)$) are holomorphic and are not identically zero on $\mathcal{D}_2$. We can always find a path of the integration in eq. (5.16) on which $M^b(v,1)^{-1}$ exists. The third comment is that the right-hand-side of eq. (5.16) (or eq. (5.14)) depends on the flavor index b (that is implicit in $U(v, v')$ and $X(v')$), while the left-hand-side is independent of it. This implies that the initial values $\hat{D}'^{bd}(1)$ are not freely chosen (see below). The final comment is that the expression (5.16) is not necessarily holomorphic and will not be, in particular, single-valued. This is indeed the case for a finite number of flavors as we will see below.

In a naive continuum limit, $(D\phi)^d_n$ will be expanded as follows:[4]

$$\frac{1}{a}(D\phi)^d_n \longrightarrow \frac{1}{a}\sum_{b=1}^{N}\hat{D}^{bd}(1)\phi^b(x) + \sum_{b=1}^{N}\hat{D}'^{bd}(1)\partial_x\phi^b(x) + O\big(a\partial_x^2\phi(x)\big)\,, \tag{5.19}$$

where a is a lattice spacing. We expect that the difference operator becomes a first-order derivative in the naive continuum limit. This requires, in addition to eq. (5.13), that $\hat{D}'^{bd}(1)$ should be non-vanishing and possess N independent eigenstates with respect to the flavor indices, otherwise the difference operator could not act on N-flavor fields properly. This observation guarantees that $\hat{D}'^{bd}(1)$ can be written in a diagonal form $\delta^{b,d}\beta^b$ by choosing an appropriate flavor basis. Then, it follows from eq. (5.14) at $v = 1$ and eq. (5.13) that β^b should be flavor-independent, i.e.

$$\hat{D}'^{bd}(1) = \delta^{b,d}\beta\,. \tag{5.20}$$

Inserting eq. (5.20) into eq. (5.16), we finally find

$$\hat{D}^{ab}(v) = \delta^{a,b}\beta\log v\,. \tag{5.21}$$

This is a straightforward extension of the one-flavor case (4.5). Since the logarithmic function is not holomorphic on $\mathcal{D}_1$ and is not local, we thus arrive at the no-go theorem 3, as promised.

Before closing this section, it may be instructive to discuss some properties of the product rule. If the flavor indices and lattice sites of the product rule are mutually independent, $M^b(v, w)$ are written as $M^b(v, w) = M^b(1, 1)F(v, w)$, where the complex function $F(v, w)$ has no flavor index.[5] It immediately follows that the first term on the right-hand-side of eq. (5.14) vanishes and the second term becomes independent of v. We then find $\hat{D}^{ab}(v)$ to be proportional to $\log v$, as found in eq. (5.21). Thus we may conclude, from the observations in this section, that at least a nontrivial connection between flavors and lattice sites for both of the product rule and the difference operator is necessary in order to realize a Leibniz rule to escape from our no-go theorem somehow in a local lattice field theory. This is the purpose of the next section.

6. Matrix representation

We have proved a no-go theorem about the existence of a product rule and a difference operator in any local lattice field theories of finite flavors. A way to escape from the no-go

[4] Eq. (5.19) will be obtained by noting that $\phi^b_m \equiv \phi^b(ma) = e^{ma\partial_x}\phi^b(x)|_{x=0}$.

[5] If we further require the associative law, $F(v, w)$ is shown to be of the form $F(v, w) = F(vw)/F(v)F(w)$ with a holomorphic function $F(v)$. This implies that $M^b(v, w)$ can reduce to $M^b(1, 1)$ by local field redefinitions.

theorem is to consider infinite flavor systems with a nontrivial connection between lattice sites and flavors. The proof given in the previous section cannot be applied for an infinite number of flavors since the holomorphy/locality of the product rule and the difference operator is not necessarily preserved in diagonalizing $D'^{ab}(1)$ by field redefinitions. This is because there is no guarantee that a linear combination of an infinite number of holomorphic functions in general is still holomorphic, though it is true for any linear combination of a finite number of holomorphic functions. A nontrivial connection between lattice sites and flavors is also inevitable because without any nontrivial relations the solution to eq. (5.14) would be proportional to the non-holomorphic function $\log v$, irrespective of the size of flavors, as discussed in the previous section.

As a candidate, we present a representation of matrix fields as lattice fields. A matrix Φ_{ij} is identified with a lattice field ϕ^a_n. The correspondence of the indices are given by $a = i - j$ and $n = i + j$. A product of two matrices, $(\Phi\Psi)_{ij}$, leads to a product rule for lattice fields, $(\phi \cdot \psi)^c_n$, as

$$C^{abc}_{lmn} = \delta_{l-n,b}\delta_{n-m,a}\delta_{c,a+b} \,. \tag{6.1}$$

It should be emphasized that the coefficients C^{abc}_{lmn} give a nontrivial connection between lattice sites and flavor indices and that the number of flavors has to be infinite with an infinite lattice volume. Since C^{abc}_{lmn} are translationally invariant and local, we can define holomorphic functions as

$$(M^b(v,w))_{ac} \equiv \hat{C}^{abc}(v,w) = \delta_{a+b,c} v^b w^{-a}. \tag{6.2}$$

It follows from eq. (5.16) that we find the expression

$$\hat{D}^{ac}(v) = \begin{cases} \frac{\hat{D}'^{\,b,b-a+c}(1)}{2(a-c)}(v^{a-c} - v^{-a+c}), & \text{for } a \neq c\,, \\ \\ \hat{D}'^{\,b,b}(1)(\,\log v + i2\pi\omega), & \text{for } a = c\,, \end{cases} \tag{6.3}$$

where ω is an integer which would come from the ambiguity how to choose the path of the integration in eq. (5.16). To avoid the logarithmic singularity, we have to impose the conditions

$$\hat{D}'^{\,b,b}(1) = 0 \quad \text{for } b = 1, 2, \cdots, N\,. \tag{6.4}$$

We further require that $\hat{D}'^{\,ac}(1)$ depend only on $a - c$. This is because the right-hand-side of eq. (6.3) should be independent of the index b. Thus, we have succeeded to obtain the difference operator $\hat{D}^{ac}(v)$ to be of the form[6]

$$\hat{D}^{ac}(v) = \tilde{d}(a-c)\,(v^{a-c} - v^{-a+c})\,, \tag{6.5}$$

where $\tilde{d}(a-c) \equiv \hat{D}'^{\,b,b-a+c}(1)/2(a-c)$. Since $\hat{D}^{ac}(v)$ are holomorphic functions, we can have a translationally invariant local difference operator D^{ac}_{mn} satisfying the Leibniz rule as

$$D^{ac}_{mn} = \tilde{d}(a-c)\,(\delta_{m-n,a-c} - \delta_{m-n,-a+c})\,. \tag{6.6}$$

[6]It will be instructive to note that $\hat{D}^{ac}(v)$ (or $\hat{D}'^{\,ac}(1)$) given in eq. (6.5) could be diagonalized by field redefinitions only if we allow the product rule and the difference operator to be non-holomorphic/non-local.

Although we can directly verify that the difference operator (6.6) and the product rule (6.1) satisfy the Leibniz rule (5.3) with infinite flavors, it is more transparent in a matrix representation, where the difference operator can be represented as the commutator such that

$$(D\phi)_n^b \equiv [\, d, \Phi\,]_{ij} \tag{6.7}$$

with the identification of $b = i - j, n = i + j$ and $d_{ij} = \tilde{d}(j - i)$. Then, it is not difficult to see that the Leibniz rule

$$D(\phi \cdot \psi) = (D\phi) \cdot \psi + \phi \cdot (D\psi) \tag{6.8}$$

is replaced by

$$[\, d, \Phi\Psi\,] = [\, d, \Phi\,]\Psi + \Phi[\, d, \Psi\,]\,, \tag{6.9}$$

which is rather a trivial relation. In addition, we note that the product rule satisfies the associative law

$$\sum_j \sum_d C^{abd}_{lmj}\, C^{dce}_{jnk} = \sum_j \sum_d C^{ade}_{ljk}\, C^{bcd}_{mnj}\,, \tag{6.10}$$

which is also trivially satisfied in the matrix representation, as $\Phi(\Psi\Lambda) = (\Phi\Psi)\Lambda$. It is interesting to note that this matrix representation has already been applied to a quantum mechanical supersymmetric lattice model [18], where the lattice version of the full supersymmetry is realized.

7. Summary and discussions

We have first shown the no-go theorem for general one-flavor systems that it is impossible to construct a lattice theory in an infinite lattice volume with a product rule and a difference operator that satisfy the following three properties: (i) translation invariance, (ii) locality and (iii) Leibniz rule. It turns out that the theorem holds even for multi-flavor systems. Our proof of the theorem shows that any difference operator satisfying the Leibniz rule can be determined from information of the product rule and some initial data. In fact, no difference operator is found to be local for general multi-flavor systems. If we allow the difference operator to be non-local, we can construct it through the equation (5.16).

A breakthrough to evade the no-go theorem is to consider an infinite number of flavors and a nontrivial connection between lattice sites and flavors. We presented such a lattice theory that infinite-flavor fields are defined from matrix fields. The product rule of two fields is just the product of two matrices and the difference operator that satisfies the Leibniz rule is found to be a commutator with a matrix d. They are all local and translationally invariant with respect to lattice sites. Furthermore, the product rule turns out to satisfy the associative law. In our forthcoming papers, we shall discuss the lattice theory equipped with the above tools and clarify their properties, in detail. In particular, we shall present lattice supersymmetric models which realize the lattice version of the full supersymmetry.

Another way we could incorporate an infinite number of flavors is to consider extra dimensions where locality or translational invariance in the whole space would be somehow broken whereas those of the target space should be preserved. We also hope to report some attempts in this direction elsewhere.

Other possibilities to escape from the no-go theorem may be to generalize translational invariance and locality. A candidate is a lattice formulation based on non-commutative geometry [19] . Further analysis in translational invariance and locality should be done.

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A. Factorization of the product rule

In this appendix, we prove the relation (3.3).

Setting $v = 1$ in eq. (3.2) leads to the relation $\hat{C}(1,w)\hat{C}(w,z) = \hat{C}(1,wz)\hat{C}(w,z)$, which reduces to $\hat{C}(1,w) = \hat{C}(1,wz)$ on the 2-dimensional complex domain $\mathcal{F}_2 = \{(w,z)|\,\hat{C}(w,z) \neq 0\,\}$. This implies that $\hat{C}(1,w)$ is independent of w on $\mathcal{D}_1$. Setting $v = 1/w$ in eq. (3.2) leads to the relation $\hat{C}(1/w,wz) = \hat{C}(1/w,w)\hat{C}(1,z)/\hat{C}(w,z)$ on $\mathcal{F}_2$. If $\hat{C}(1,z)$ is zero, then $\hat{C}(1/w,wz)$ would be identically zero. However, this cannot be the case for a nontrivial product rule $\hat{C}(v,w)$. Therefore, $\hat{C}(1,z)$ cannot be zero. This argument is applicable for $\hat{C}(v,1)$. We thus conclude

$$\hat{C}(1,z) = \hat{C}(v,1) = \alpha\,, \tag{A.1}$$

where α is a nonzero constant.

By differentiating eq. (3.2) with respect to v and then taking $v = 1$, we have

$$f(w)\hat{C}(w,z) + \alpha w\partial_w\hat{C}(w,z) = f(wz)\hat{C}(w,z)\,, \tag{A.2}$$

where $f(w) \equiv \partial_v\hat{C}(v,w)|_{v=1}$. We regard eq. (A.2) as a differential equation for $\hat{C}(w,z)$ with a given function $f(w)$. With the initial condition (A.1), the differential equation (A.2) can easily be solved as

$$\hat{C}(w,z) = \alpha\exp\left\{\frac{1}{\alpha}\int_1^w\frac{du}{u}\Big(f(uz) - f(u)\Big)\right\}. \tag{A.3}$$

We should emphasize that the expression (A.3) is well-defined and that there is no ambiguity in the definition of the integral with respect to u because

$$\oint\frac{du}{u}\Big(f(uz) - f(u)\Big) = 0 \tag{A.4}$$

for any closed loop on $\mathcal{D}_1$.

Since $f(u)$ is holomorphic on $\mathcal{D}_1$, it can be uniquely expanded in the Laurent series as

$$f(u) = \sum_{n=-\infty}^{\infty} f_n\,u^n. \tag{A.5}$$

Let us introduce a new holomorphic function $\tilde{f}(u)$ as

$$\tilde{f}(u) \equiv f(u) - f_0 \,. \tag{A.6}$$

Then, for any closed loop on $\mathcal{D}_1$ we find

$$\oint \frac{du}{u} \tilde{f}(u) = \oint \frac{du}{u} \sum_{n \neq 0} f_n \, u^n = 0 \,. \tag{A.7}$$

In terms of $\tilde{f}(u)$, eq. (A.3) can be rewritten as

$$\begin{aligned}
\hat{C}(w,z) &= \alpha \exp \left\{ \frac{1}{\alpha} \int_1^w \frac{du}{u} \Big(f(uz) - f(u) \Big) \right\} \\
&= \alpha \exp \left\{ \frac{1}{\alpha} \int_1^w \frac{du}{u} \Big(\tilde{f}(uz) - \tilde{f}(u) \Big) \right\} \\
&= \frac{\frac{1}{\alpha} \exp \left\{ \frac{1}{\alpha} \int_1^{wz} \frac{du}{u} \tilde{f}(u) \right\}}{\frac{1}{\alpha} \exp \left\{ \frac{1}{\alpha} \int_1^{w} \frac{du}{u} \tilde{f}(u) \right\} \frac{1}{\alpha} \exp \left\{ \frac{1}{\alpha} \int_1^{z} \frac{du}{u} \tilde{f}(u) \right\}} \\
&\equiv \frac{F(wz)}{F(w)F(z)} \,.
\end{aligned} \tag{A.8}$$

We note that the function $F(w)$ is well-defined and that there is no ambiguity for the integral because of the property (A.7).

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